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Flexural-torsional-coupled vibration analysis of Euler-Bernoulli beam by using the differential transform method

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Abstract

This study introduces an analytical solution method which is called the differential transform method (DTM) to analyze the free vibration response of a beam which features material coupling between flapwise bending and torsional vibrations. The governing differential equations of motion are solved by applying DTM. The natural frequencies are obtained and compared with a study taken form the literature. A good agreement is found between the results of this study and the results of the study taken from the literature.

Key words: Differential transform method, bending-torsion coupling effect, free vibration

1. Introduction

The free vibration analysis of a bending-torsion coupled beam is increasingly being investigated in several engineering applications including rotating and non-rotating machinery, aerospace structures, windmills, etc. The dynamic characteristics, i.e., natural frequencies and related mode shapes obtained in the free vibration analysis, of these structures is used in the design and performance evaluation in engineering applications because they are required to determine resonant responses and to perform forced vibration analysis.

Solving the extended aeroelasticity problem, which often arises in aviation, requires explicit choice of the structural model for the aircraft. As a first approximation [1], the fuselage, wing, and both the horizontal and vertical stabilizers in the empennage are modeled as beams clamped at the origin of the respective body axes and undergoing bending and torsion. For this reason, the effect of coupling between the bending and torsional vibrations, which can occur in both solid and thin-walled beams, is of particular interest from an aeroelastic standpoint [2]. Eslimy-Isfahany, Banerjee and Sobey [3] analysed free and forced vibration of bending-torsion coupled beam with Euler-Bernoulli beam theory for deterministic and random loads using modal analysis. Hashemi and Richard [4] worked on the study of the bending and torsion vibration response of the axially loaded beam by using dynamic finite element method in order to obtain natural frequencies and related mode shapes for uncoupled and coupled beam, separately. Surace [5] used a new approximate method, based on the use of Green functions, for the analysis of the modal characteristics of non-

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uniform blades corresponding to the coupled flapwise bending, chordwise bending and torsion of both rotating and non-rotating blades. Liu and Shu [6] used a method, called as a closed form solution, to solve analytically frequencies and corresponding mode shapes of coupled bending-torsion vibration model with single delamination subjected to axial loads and static end moments. Oh and Yoo [7] examined the coupling effects between stretching, bending, and torsion of a rotating pre-twisted blade using a complex modal analysis method. These studies have shown [3-7] that compared with numerical methods, analytical methods are more preferred to solve vibration problem in order to obtain the behavior of coupled bending-torsion beams.

One of the analytical methods for the solution of differential equations occurred in the engineering applications is the differential transform method (DTM). The concept of this method was first introduced by Zhou [8] in 1986 and used for a study regarding with electrical circuits. Based on an analytical approach, the DTM has an ability to solve linear and nonlinear equations [9], ordinary [10] and partial [11] differential equations, integro-differential [12] and fractional differential [13] equations. One of the first researchers who used DTM, Hassan [14] investigated the solution of Sturm-Liouville eigenvalue problem using DTM and compared the results with analytical results that were obtained by other methods. Ho and Chen [15] utilized the same method for analysis of general elastically end restrained non-uniform beams. As the method became popular, it has been recently become a useful and practical solution technique to apply it in the vibration analysis of uniform or non-uniform beams [16] and plates [17]. Since, in the studies [8-17], it has been shown that the DTM is an efficient tool to solve differential equations with an analytical point of view, it has still gained much attention of several researchers in the structural engineering problems.

In this study, vibration analysis of Euler-Bernoulli type beam with bending-torsion coupling is performed using the differential transform method (DTM). The first six natural frequencies of the beam are found and the results are compared with an illustrative example taken from the literature.

2. Formulation and Method

2.1. Fomulation

The governing partial differential equation of motion is derived for the out-of-plane free vibration of a tapered cantilever Euler-Bernoulli type bending-torsion coupled beam as an aircarft wing, as shown in Fig 1(a). The allowable displacements consists of a flexural translation $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ in the $\boldsymbol{z}-$direction and a torsional rotation $\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{t})$ about the $\boldsymbol{x}-$axis; where $\boldsymbol{x}$ and $\boldsymbol{t}$ denote the distance from origin and time, respectively. The cross-sectional loads are represented by a force per unit length $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{t})$ acting parallel to $\boldsymbol{z}-$axis and applied through the shear center, together with a torque per unit length $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{t})$ about $\boldsymbol{x}-$axis as shown in Fig 1(b).

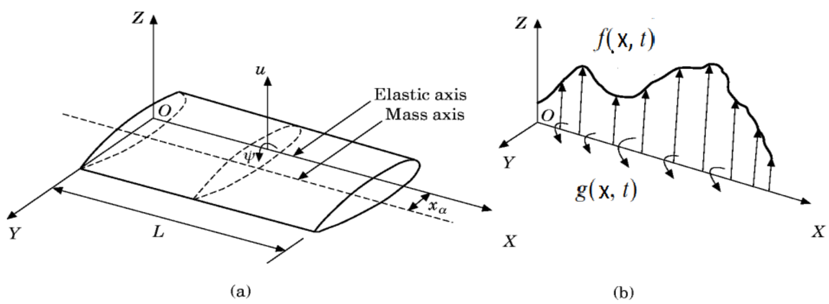

Fig 1. The coordinate system and notation for coupled bending-torsional vibration of a beam, (b) The distributions of bending and torsional loads [3].

According to the Euler - Bernoulli beam theory, the governing differential equation of motion for the out-of-plane coupled bending and torsion motion is as follows

$$EI\frac{\partial^4 u}{\partial x^4}+m\frac{\partial^2 u}{\partial t^2}+c_1\left(\frac{\partial u}{\partial t}-x_\alpha\frac{\partial \psi}{\partial t}\right)-mx_\alpha\frac{\partial^2 \psi}{\partial t^2}=f(x,t) \quad (1)$$

$$GJ\frac{\partial^2 \psi}{\partial x^2}+mx_\alpha\frac{\partial^2 u}{\partial t^2}-I_\alpha\frac{\partial^2 \psi}{\partial t^2}-c_2\frac{\partial \psi}{\partial t}+c_1x_\alpha\frac{\partial u}{\partial t}=g(x,t) \quad (2)$$

where EI and GJ are, respectively, the bending and torsional rigidity, m is the mass per unit length and I_α is the mass moment of inertia per unit length about the $x-axis$ of the beam. The coeeficints c_1 and c_2 are viscous damping terms per unit length in flexure and torsion, respectively.

The boundary conditions for a cantilever Euler – Bernoulli beam with coupled bending – torsion motion can be expressed as follows

$$u(x,t)=\frac{\partial u(x,t)}{\partial x}=0, \qquad at \quad x=0 \quad (3a)$$

$$\psi(x,t)=0, \qquad at \quad x=0 \quad (3b)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^3 u(x,t)}{\partial x^3}=0, \qquad at\ x=L \quad (3c)$$

$$\frac{\partial \psi(x,t)}{\partial x}=0, \qquad at\ x=L \quad (3d)$$

2.2. Method

The Differential Transform Method (DTM) is a transformation technique based on a power series expansion and it is an analytical method to obtain analytical solutions of the differential equations. In this method, certain transformation rules are applied and the governing differential equations and boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem with great accuracy.

Consider a function $\boldsymbol{f(x)}$ which is analytic in a domain $\boldsymbol{D}$ and let $\boldsymbol{x = x_0}$ represent any point in $\boldsymbol{D}$. The function $\boldsymbol{f(x)}$ is then represented by a power series whose center is located at $\boldsymbol{x_0}$. The differential transform of the function $\boldsymbol{f(x)}$ is given by

$$\boldsymbol{F[k]} = \frac{\mathbf{1}}{\boldsymbol{k}!}\left(\frac{\boldsymbol{d^k f(x)}}{\boldsymbol{dx^k}}\right)_{x=x_0} \qquad (\mathbf{4})$$

where $\boldsymbol{f(x)}$ is the original function and $\boldsymbol{F[k]}$ is the transformed function.

The inverse transformation is defined as

$$\boldsymbol{f(x)} = \sum_{\boldsymbol{k=0}}^{\infty} (\boldsymbol{x - x_0})^{\boldsymbol{k}} \boldsymbol{F[k]}. \qquad (\mathbf{5})$$

The theorems that are frequently used in the transformation of the differential equations and the boundary conditions are introduced in Table1.

Original function	**Transformed Functions**
$\boldsymbol{f(s) = g(s) \pm h(s)}$	$F_D\{k\} = G_D\{k\} \pm H_D\{k\}$
$\boldsymbol{f(s) = \alpha\, g(s)} \quad (\boldsymbol{\alpha} \in \mathbb{R})$	$F_D\{k\} = \alpha G_D\{k\} \quad (\alpha \in \mathbb{R})$
$\boldsymbol{f(s)} = \frac{\boldsymbol{d^m g(s)}}{\boldsymbol{ds^m}}$	$F_D\{k\} = (k+1)(k+2) + \cdots + (k+m)G_D\{k+m\}$
$\boldsymbol{f(s) = exp(\alpha s)} \quad (\boldsymbol{\alpha} \in \mathbb{R})$	$F_D\{k\} = \frac{\alpha^k}{k!} \quad (\alpha \in \mathbb{R})$
$\boldsymbol{f(s) = s^m}$	$F_D\{k\} = \delta(k-m) = \begin{cases} 1, & if\ \ k = m \\ 0, & if\ k \neq m \end{cases}$
$\boldsymbol{f(s) = g(s)h(s)}$	$F_D\{k\} = \sum_{l=0}^{k} G_D\{l\}\, H_D\{k-l\}$

Table 1. DTM theorems for DTM.

By applying the rules defined in Table 1 and Table 2 to Eq(1) and Eq(2) for the undamped free vibration analysis, i.e. when $\boldsymbol{f(x,t) = g(x,t) = c_1 = c_2 = 0}$, the following analytical expressions are obtained.

$$\boldsymbol{U}[\boldsymbol{k}+\mathbf{4}] = \frac{\boldsymbol{a}\boldsymbol{U}[\boldsymbol{k}] - \boldsymbol{c}\boldsymbol{\Psi}[\boldsymbol{k}]}{(\boldsymbol{k}+\mathbf{1})(\boldsymbol{k}+\mathbf{2})(\boldsymbol{k}+\mathbf{3})(\boldsymbol{k}+\mathbf{4})} \tag{6}$$

$$\boldsymbol{\Psi}[\boldsymbol{k}+\mathbf{2}] = \frac{\boldsymbol{d}\boldsymbol{U}[\boldsymbol{k}] - \boldsymbol{b}\boldsymbol{\Psi}[\boldsymbol{k}]}{(\boldsymbol{k}+\mathbf{1})(\boldsymbol{k}+\mathbf{2})} \tag{7}$$

with boundary conditions

$$\boldsymbol{U}[\mathbf{0}] = \boldsymbol{U}[\mathbf{1}] = \boldsymbol{\Psi}[\mathbf{0}] = \mathbf{0} \qquad \boldsymbol{at} \;\; \boldsymbol{\xi} = \mathbf{0} \tag{8}$$

$$\sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} \boldsymbol{k}(\boldsymbol{k}-\mathbf{1})\boldsymbol{U}[\boldsymbol{k}] = \mathbf{0}, \;\; \sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} \boldsymbol{k}(\boldsymbol{k}-\mathbf{1})(\boldsymbol{k}-\mathbf{2})\boldsymbol{U}[\boldsymbol{k}] = \mathbf{0}, \;\; \sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} \boldsymbol{k}\boldsymbol{\Psi}[\boldsymbol{k}] = \mathbf{0}, \;\; \boldsymbol{at} \;\; \boldsymbol{\xi} = \mathbf{1}. \tag{9}$$

3. Results

A typical cantilever aircarft wing, in which flexural and torsional motions are coupled, is defined by the following data [4]: (i) bending rigidity $(EI) = 9.75 \times 10^6 Nm^2$; (ii) torsional rigidity $(GJ) = 9.88 \times 10^5 Nm^2$; (iii) mass per unit length $(m) = 35.75\ kg/m$; (iv) mass moment of inertia per unit length $(I_\alpha) = 8.65\ kgm$; (v) distance between mass center and shear center $(x_\alpha) = 0.18\ m$; (vi) length of the wing $(L) = 6\ m$.

The results of the natural frequencies of the coupled bending-torsion beam are shown in Table 2.

Natural freq. (rad/sec)	DTM	Ref. [4]	Error (%)
ω_1	49.61	49.62	0.01
ω_2	97.04	97.05	0.01
ω_3	248.87	249.00	0.13
ω_4	355.59	357.54	1.95
ω_5	451.45	452.57	1.12
ω_6	610.32	610.63	1.31

Table 2. Natural frequencies of the beam for coupled and uncoupled, separately, bending-torsion behaviour.

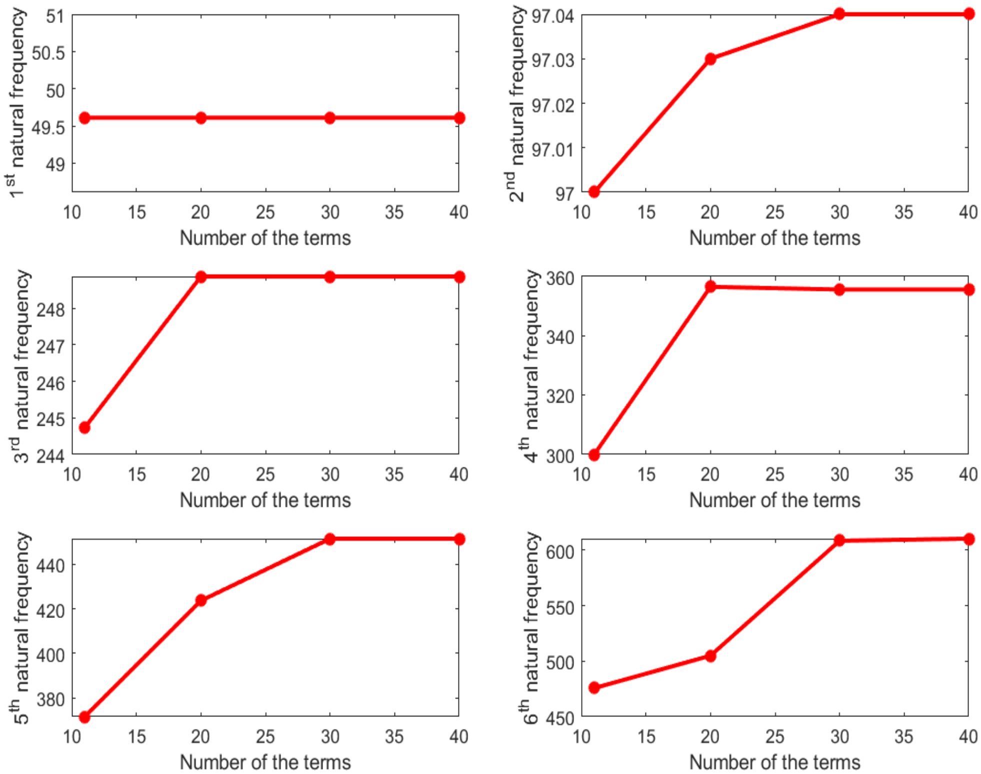

Fig. 2. Convergence of the first six natural frequencies of the typical cantilever aircraft wing.

Regarding with the convergence of the first six natural frequencies, Fig. 2 shows the number of terms included in DTM versus natural frequencies. In this figure, it is seen that in order to evaluate up to sixth natural frequency with two-digit precision, it is necessary to include 30 terms. Additionally, it should be noted that higher modes appear when more terms are taken into account in DTM application.

In Fig. 3, the mode shapes of the wing are given. Fig. 3 reveals that all normal modes are the coupled modes; however, in the second normal mode bending motion is very little when it is compared with the torsion motion, especially towards the wing tip.

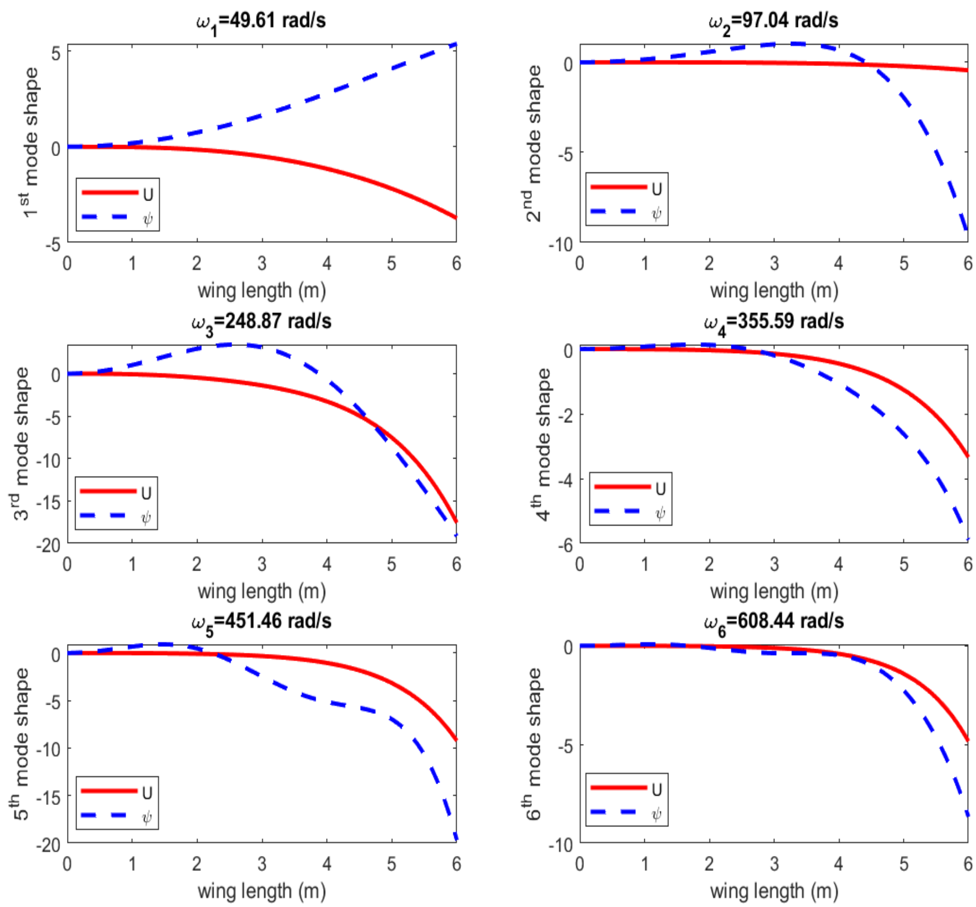

Fig 3. The normal mode shapes of the cantilever aircraft wing with bending-torsion coupling.

Conclusions

In this study, free vibration analysis of Euler-Bernoulli type beam with bending-torsion coupling and uncoupling, respectively, is performed using an analytical method, called the differential transform method (DTM). The first six natural frequencies have been found and the results have been compared with the results of Ref. [4]; an agreement with fairly good accuracy is found between the studies. Furthermore, in order to obtain the first six natural frequencies with the minimum errors, it is necessary to include the first 30 terms in DTM. Beside these, the mode shapes of the wing are obtained and interpreted in trems of coupling behavior.

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